

Conditional regularity of solutions of the three dimensional Navier-Stokes equations and implications for intermittency

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Two unusual time-integral conditional regularity results are presented for the three-dimensional Navier-Stokes equations. The ideas are based on L^{2m} -norms of the vorticity, denoted by $\Omega_m(t)$, and particularly on $D_m = \Omega_m^{\alpha_m}$, where $\alpha_m = 2m/(4m-3)$ for $m \geq 1$. The first result, more appropriate for the unforced case, can be stated simply: if there exists an $1 \leq m < \infty$ for which the integral condition is satisfied ($Z_m = D_{m+1}/D_m$)

$$\int_0^t \ln \left(\frac{1 + Z_m}{c_{4,m}} \right) d\tau \geq 0,$$

then no singularity can occur on $[0, t]$. The constant $c_{4,m} \searrow 2$ for large m . Secondly, for the forced case, by imposing a critical *lower* bound on $\int_0^t D_m d\tau$, no singularity can occur in $D_m(t)$ for *large* initial data. Movement across this critical lower bound shows how solutions can behave intermittently, in analogy with a relaxation oscillator. Potential singularities that drive $\int_0^t D_m d\tau$ over this critical value can be ruled out whereas other types cannot.

I. INTRODUCTION

The twin related themes of this paper are firstly the regularity problem for solutions of the three-dimensional incompressible Navier-Stokes equations and secondly the intermittent behaviour of these solutions. Traditionally the first problem, which still remains tantalizingly open, has lain in the domain of the analyst, whereas the phenomenon of intermittency has tended to be more of interest to the physics and engineering fluid dynamics communities. It will be demonstrated in this paper that these two issues are intimately related & require simultaneous study.

A. History

Formally, a weak solution $\mathbf{u}(\mathbf{x}, t)$ of the three-dimensional Navier-Stokes equations

$$(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p + \mathbf{f}(\mathbf{x}), \quad (\text{I.1})$$

with $\text{div } \mathbf{u} = 0$, is called regular if the H_1 -norm is continuous [1]. What is commonly referred to as ‘conditional regularity’ can be achieved if it is found necessary to impose assumptions on certain system variables such as the velocity field. The early work of Prodi [3], Serrin [4] and Ladyzhenskaya [5] can be summarized thus [6, 7]: every Leray-Hopf solution of the incompressible three-dimensional Navier-Stokes equations with $\mathbf{u} \in L^r((0, T); L^s)$ is regular on $(0, T]$ provided $2/r + 3/s = 1$, with $s \in (3, \infty]$, or if $\mathbf{u} \in L^\infty((0, T); L^p)$ with $p > 3$. The long-standing case $s = 3$ was finally settled by von Wahl [8] and Giga [9] who proved regularity in the space $C((0, T); L^3)$: see also Kozono and Sohr [10] and Escauriaza, Seregin and Sverák [11]. In summary, the $s = 3$ case seems tantalizingly close to the bounded case $s = 2$, but not quite close enough. More recent results exist where conditions are imposed on either the pressure or on one derivative of the velocity field: see the references in Kukavica and Ziane [12, 13], Zhou [14], Cao & Titi [15, 16], Cao [17], Cao, Qin and Titi (for channel flows) [18], Chen and Gala [19] and the review by Doering [20]. Results on the direction of vorticity can be found in Constantin and Fefferman [21] and Vasseur [22], and those on the use of Besov spaces in Cheskidov and Shvydkoy [23].

Finally, in recent work, Biswas and Foias [24] have considered analyticity properties of Navier-Stokes solutions in which they have studied the maximal space analyticity radius associated with a regular solution involving Gevrey-class norms. Intermittency properties have also been studied by Grujic [25] and Dascalu and Grujic [26, 27] using methods very different from those employed in this paper.

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B. Motivation and notation

In an entirely different thread of intellectual endeavour, the seminal experimental paper of Batchelor and Townsend [28] investigated the phenomenon of intermittency in wind tunnel turbulence by observing that the flatness of their signals (the ratio of the fourth order moment to the square of the second order moment) took much higher values than those expected for random Gaussian behaviour. They concluded that the vorticity is neither spatially nor temporally distributed in an even fashion but undergoes local clustering or spottiness, which is consistent with the appearance of spikes in the signals interspersed by longer quiescent periods. This is now considered to be a classic characteristic of intermittency. These ideas have been developed and extended in many subsequent experiments and computations: see the papers by Kuo and Corrsin [29], Sreenivasan [30], Meneveau and Sreenivasan [31] and the books by Frisch [32] and Davidson [33] for further references.

Most of these discussions have been based around Kolmogorov's statistical theory with the widespread use of velocity structure functions to study intermittent behaviour. However, structure functions are not easily translatable into results in Navier-Stokes analysis (see some of the arguments in Kuksin [34] and Dascaliuc and Grujic [27]). The main difficulty lies in translating the special conditions needed to prove regularity listed in §IA into sensible physics while conversely making sense of the experimental observations in terms of Navier-Stokes variables. The two threads can be merged if the spiky nature of the vorticity field is considered in the context of L^p -norms of the vorticity $\omega = \text{curl } \mathbf{u}$ with $p = 2m$

$$\Omega_m(t) = \left(L^{-3} \int_V |\omega|^{2m} dV \right)^{1/2m} + \varpi_0, \quad m \geq 1, \quad (\text{I.2})$$

on a periodic box $[0, L]^3$. The additive frequency $\varpi_0 = \nu L^{-2}$ is present for technical reasons. Clearly the $\Omega_m(t)$ are ordered for all t such that

$$\varpi_0 \leq \Omega_1(t) \leq \Omega_2(t) \leq \dots \leq \Omega_m(t) \leq \Omega_{m+1}(t) \dots, \quad (\text{I.3})$$

where $L^{3/2}\Omega_1(t)$ is the H_1 -norm. Control from above on any one of the Ω_m will also control the H_1 -norm from above which is the ultimate key to regularity.

The Navier-Stokes equations have a well-known invariance property under the transformations $x' = \epsilon x$, $t' = \epsilon^2 t$, $u = \epsilon u'$ and $p = \epsilon^2 p'$. Under these transformations Ω_m scales as

$$\Omega_m^{\alpha_m} = \epsilon \Omega_m'^{\alpha_m}, \quad \alpha_m = \frac{2m}{4m-3}. \quad (\text{I.4})$$

Thus it is natural to define

$$D_m(t) = [\varpi_0^{-1} \Omega_m(t)]^{\alpha_m}. \quad (\text{I.5})$$

While the norms Ω_m are ordered with increasing m as in (I.3), the α_m decrease with m . Thus there is no natural ordering among the D_m . Note also that D_1 is the square of the H_1 -norm.

For the forced case, the dimensionless Grashof number Gr is based on the bounded-ness of the root-mean-square $f_{rms}^2 = L^{-3} \|\mathbf{f}\|_2^2$ of the divergence-free forcing $\mathbf{f}(\mathbf{x})$ and is defined as

$$Gr = \frac{L^3 f_{rms}}{\nu^2}. \quad (\text{I.6})$$

C. Two fundamental results

There are two results that form the basis of those given in later sections. The first is a theorem on time integrals or averages of D_m [35] which uses a result of Foias, Guillopé and Temam [36]. This proof will not be repeated:

Theorem 1 For $1 \leq m \leq \infty$, and α_m defined as $\alpha_m = \frac{2m}{4m-3}$, weak solutions obey

$$\int_0^t D_m d\tau \leq c (t Gr^2 + \eta_1), \quad (\text{I.7})$$

where $\eta_1 = L\nu^{-3}E_0$ and E_0 is the initial energy. In the unforced case the right hand side is just $c\eta_1$.

In [35] this result was converted into a set of length scales. Let the time average up to time T be defined by

$$\langle D_m \rangle_T = \limsup_{D(0)} \frac{1}{T} \int_0^T D_m(\tau) d\tau \quad (\text{I.8})$$

in which case (I.7) can be re-expressed as

$$\langle D_m \rangle_T \leq cGr^2 + O(T^{-1}) . \quad (\text{I.9})$$

Then, motivated by the definition of the Kolmogorov length for $m = 1$, a set of length scales can be defined thus :

$$(L\lambda_m^{-1})^{2\alpha_m} := \langle D_m \rangle_T . \quad (\text{I.10})$$

In [35] the bounds in (I.9) were expressed as Re^3 instead of Gr^2 based on a device of Doering and Foias [37] who used the square of the averaged velocity $U_0^2 = L^{-3} \langle \|\mathbf{u}\|_2^2 \rangle_T$ to define the Reynolds number $Re = U_0 L \nu^{-1}$; for Navier-Stokes solutions this leads to the inequality $Gr \leq cRe^2$. However, the preference in this paper is to remain with the Grashof number Gr . In terms of Re equation (I.10) becomes

$$L\lambda_m^{-1} \leq cRe^{\frac{3}{2\alpha_m}} . \quad (\text{I.11})$$

When $m = 1$, $\alpha_1 = 2$, and thus $L\lambda_1^{-1} \leq cRe^{3/4}$, which is consistent with Kolmogorov's statistical theory [32].

The second result is a differential inequality for the D_m . Any attempt to time-differentiate the vorticity field creates problems because only weak solutions exist. Circumvention of this difficulty requires a contradiction strategy commonly used in geometric analysis: assume that there is a maximal interval of existence and uniqueness $[0, T^*)$ which, for the three-dimensional Navier-Stokes equations, implies that $H_1(T^*) = \infty$. In any subsequent calculation, if the H_1 -norm were to turn out bounded in the limit $t \rightarrow T^*$, then a contradiction would result and so the interval $[0, T^*)$ could not be maximal. Moreover, it cannot be zero, so T^* would have to be infinite.

Define three frequencies

$$\varpi_{1,m} = \varpi_0 \alpha_m c_{1,m}^{-1} \quad \varpi_{2,m} = \varpi_0 \alpha_m c_{2,m} \quad \varpi_{3,m} = \varpi_0 \alpha_m c_{3,m} , \quad (\text{I.12})$$

where the constants $c_{n,m}$ ($n = 1, 2, 3$) are algebraically increasing with m . The proof of the following theorem requires some variations on a previous result [38] and is relegated to Appendix A. The dot represents differentiation with respect to time :

Theorem 2 For $1 \leq m < \infty$ on $[0, T^*]$ the $D_m(t)$ formally satisfy the set of inequalities

$$\dot{D}_m \leq D_m^3 \left\{ -\varpi_{1,m} \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} + \varpi_{2,m} \right\} , \quad (\text{I.13})$$

where $\rho_m = \frac{2}{3}m(4m+1)$. In the forced case there is an additive term $\varpi_{3,m}GrD_m$.

II. A CONDITIONAL REGULARITY RESULT FOR UNFORCED NAVIER-STOKES

A. Integration of the D_m inequality

Theorem 2 leads to the conclusion that solutions come under control pointwise in t provided $D_{m+1}(t) \geq c_{\rho_m} D_m(t)$, where $c_{\rho_m} = [c_{1,m}c_{2,m}]^{1/\rho_m}$. The following lemma shows that a time integral version of this controls solutions :

Lemma 1 For any value of $1 \leq m < \infty$ and ε uniform in the range $0 < \varepsilon < 2$, if the integral condition is satisfied

$$\int_0^t D_{m+1}^\varepsilon d\tau \geq c_{\varepsilon, \rho_m} \int_0^t D_m^\varepsilon d\tau \quad c_{\varepsilon, \rho_m} = [c_{1,m}c_{2,m}]^{\varepsilon/\rho_m} \quad (\text{II.1})$$

then $D_m(t)$ obeys $D_m(t) \leq D_m(0)$ on the interval $[0, t]$.

Remark : The case $\varepsilon = 1$ turns (II.1) into $\int_0^t D_{m+1} d\tau \geq c_{1, \rho_m} \int_0^t D_m d\tau$ both sides of which are bounded above. However, the fact that ε can take small values suggests a logarithmic result which appears in the following theorem. The proof of lemma 1 is included within its proof.

Theorem 3 For any value of $1 \leq m < \infty$, if the integral condition is satisfied

$$\int_0^t \ln \left(\frac{1 + Z_m}{c_{4,m}} \right) d\tau \geq 0, \quad Z_m = D_{m+1}/D_m \quad (\text{II.2})$$

with $c_{4,m} = [2^{\rho_m-1} (1 + c_{1,m}c_{2,m})]^{\rho_m^{-1}}$, then $D_m(t) \leq D_m(0)$ on the interval $[0, t]$.

Remark 1: This result may serve as an alternative to the Beale-Kato-Majda theorem [39].

Remark 2: The exponent ρ_m^{-1} pulls $c_{4,m}$ down close to 2 for large m which indicates that there needs to be enough intervals of time on which $Z_m > 1$ for (II.2) to hold.

Proof: The proof of Lemma 1 is addressed first. Divide (I.13) by $D_m^{3-\varepsilon}$ and integrate to obtain

$$[D_m(t)]^{\varepsilon-2} - [D_m(0)]^{\varepsilon-2} \geq \varpi_{1,m}^{(\varepsilon)} \int_0^t \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^\varepsilon d\tau - \varpi_{2,m}^{(\varepsilon)} \int_0^t D_m^\varepsilon d\tau, \quad (\text{II.3})$$

where $\varpi_{n,m}^{(\varepsilon)} = (2 - \varepsilon)\varpi_{n,m}$. Noting that $\rho_m \geq 10/3$, a Hölder inequality then easily shows that

$$\begin{aligned} \int_0^t D_{m+1}^\varepsilon d\tau &= \int_0^t \left[\left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^\varepsilon \right]^{\frac{\varepsilon}{\rho_m}} [D_m^\varepsilon]^{\frac{\rho_m - \varepsilon}{\rho_m}} d\tau \\ &\leq \left(\int_0^t \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^\varepsilon d\tau \right)^{\frac{\varepsilon}{\rho_m}} \left(\int_0^t D_m^\varepsilon d\tau \right)^{\frac{\rho_m - \varepsilon}{\rho_m}}. \end{aligned} \quad (\text{II.4})$$

(II.3) can then be re-written as

$$D_m(t) \leq \left\{ [D_m(0)]^{\varepsilon-2} + \varpi_{1,m}^{(\varepsilon)} \frac{\left(\int_0^t D_{m+1}^\varepsilon d\tau \right)^{\rho_m/\varepsilon}}{\left(\int_0^t D_m^\varepsilon d\tau \right)^{(\rho_m - \varepsilon)/\varepsilon}} - \varpi_{2,m}^{(\varepsilon)} \int_0^t D_m^\varepsilon d\tau \right\}^{-\frac{1}{2-\varepsilon}}. \quad (\text{II.5})$$

It is clear that no sign change can occur in the denominator of (II.5) if (II.1) holds.

The proof of Theorem II.2 is now addressed. Divide (I.13) by D_m^3 and integrate to obtain

$$\begin{aligned} \frac{1}{2} ([D_m(t)]^{-2} - [D_m(0)]^{-2}) &\geq \varpi_{1,m} \int_0^t \left\{ [1 + Z_m^{\rho_m}] - \left(1 + \frac{\varpi_{2,m}}{\varpi_{1,m}} \right) \right\} d\tau \\ &\geq \frac{\varpi_{1,m}}{2^{\rho_m-1}} \int_0^t \{ [1 + Z_m]^{\rho_m} - 2^{\rho_m-1} (1 + c_{1,m}c_{2,m}) \} d\tau, \end{aligned} \quad (\text{II.6})$$

where we have used $(1 + Z_m)^{\rho_m} \leq 2^{\rho_m-1} (1 + Z_m^{\rho_m})$. Re-arranging & using Jensen's inequality

$$\frac{1}{t} \int_0^t \exp F(\tau) d\tau \geq \exp \left(\frac{1}{t} \int_0^t F(\tau) d\tau \right), \quad (\text{II.7})$$

with $F = \rho_m \ln(1 + Z_m)$, the RHS of (II.6) can be written as

$$\begin{aligned} \frac{1}{t} \int_0^t \{ [1 + Z_m]^{\rho_m} - 2^{\rho_m-1} (1 + c_{1,m}c_{2,m}) \} d\tau &= \frac{1}{t} \int_0^t \{ \exp [\rho_m \ln(1 + Z_m)] - 2^{\rho_m-1} (1 + c_{1,m}c_{2,m}) \} d\tau \\ &\geq \exp \left[\frac{\rho_m}{t} \int_0^t \ln(1 + Z_m) d\tau \right] - \exp [\rho_m \ln c_{4,m}] \end{aligned} \quad (\text{II.8})$$

and thus no zero can develop if (II.2) holds, as advertised.

In both cases if a zero cannot appear in the respective denominators then $D_m(t) \leq D_m(0)$. It follows that $\Omega_m(t)$ is bounded above and thus so is the H_1 -norm (Ω_1). ■

III. SECOND INTEGRATION OF THE D_m INEQUALITY

A. A lower bound on $\int_0^t D_m d\tau$

Inclusion of the forcing in (I.13) modifies it to

$$\dot{D}_m \leq D_m^3 \left\{ -\frac{1}{\varpi_{1,m}} \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} + \varpi_{2,m} \right\} + \varpi_{3,m} Gr D_m, \quad (\text{III.1})$$

where $\rho_m = \frac{2}{3}m(4m+1)$. To proceed, divide by D_m^3 (the case $\varepsilon = 0$ of Theorem 1) to write (III.1) as

$$\frac{1}{2} \frac{d}{dt} (D_m^{-2}) \geq X_m (D_m^{-2}) - \varpi_{2,m} \quad (\text{III.2})$$

where

$$X_m = \varpi_{1,m} \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^2 - \varpi_{3,m} Gr. \quad (\text{III.3})$$

A lower bound for $\int_0^t X_m d\tau$ can be estimated thus:

$$\begin{aligned} \int_0^t D_{m+1} d\tau &= \int_0^t \left[\left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^2 \right]^{\frac{1}{\rho_m}} D_m^{\frac{\rho_m-2}{\rho_m}} d\tau \\ &\leq \left(\int_0^t \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^2 d\tau \right)^{\frac{1}{\rho_m}} \left(\int_0^t D_m d\tau \right)^{\frac{\rho_m-2}{\rho_m}} t^{1/\rho_m} \end{aligned} \quad (\text{III.4})$$

and so

$$\int_0^t X_m d\tau \geq \varpi_{1,m} t^{-1} \frac{\left(\int_0^t D_{m+1} d\tau \right)^{\rho_m}}{\left(\int_0^t D_m d\tau \right)^{\rho_m-2}} - \varpi_{3,m} t Gr. \quad (\text{III.5})$$

(III.2) integrates to

$$[D_m(t)]^2 \leq \frac{\exp \left\{ -2 \int_0^t X_m d\tau \right\}}{[D_m(0)]^{-2} - 2\varpi_{2,m} \int_0^t \exp \left\{ -2 \int_0^\tau X_m d\tau' \right\} d\tau}. \quad (\text{III.6})$$

Let us recall that $\rho_m = \frac{2}{3}m(4m+1)$ and let us also define

$$\gamma_m = \frac{\alpha_{m+1}}{2(m^2-1)}, \quad (\text{III.7})$$

then

Theorem 4 *On the interval $[0, t]$ if there exists a value of m lying in the range $1 < m < \infty$, with initial data $[D_m(0)]^2 < C_m Gr^{\Delta_m}$, for which the integral lies on or above the critical value*

$$c_m (t Gr^{2\delta_m} + \eta_2) \leq \int_0^t D_m d\tau \quad (\text{III.8})$$

where $\eta_2 \geq \eta_1 Gr^{2(\delta_m-1)}$ and where $(1 \leq \Delta_m \leq 4)$

$$\Delta_m = 4 \{ \delta_m (1 + \rho_m \gamma_m) - \rho_m \gamma_m \} \quad \text{with} \quad \frac{1 + 4\rho_m \gamma_m}{4(1 + \rho_m \gamma_m)} < \delta_m < 1, \quad (\text{III.9})$$

then $D_m(t)$ decays exponentially on $[0, t]$.

Remark: $\delta_m \searrow 11/20$ for large m so enough slack lies between the upper and lower bounds on $\int_0^t D_m d\tau$.

Proof: It is not difficult to prove that $\Omega_m^{m^2} \leq \Omega_{m+1}^{m^2-1} \Omega_1$ for $m > 1$, from which it is easily found that

$$\frac{\int_0^t D_{m+1} d\tau}{\int_0^t D_m d\tau} \geq \left(\frac{\int_0^t D_m d\tau}{\int_0^t D_1 d\tau} \right)^{\gamma_m}. \quad (\text{III.10})$$

Thus (III.5) can be re-written as

$$\begin{aligned} \int_0^t X_m d\tau &\geq \varpi_{1,m} t^{-1} \frac{\left(\int_0^t D_m d\tau \right)^{\rho_m \gamma_m + 2}}{\left(\int_0^t D_1 d\tau \right)^{\rho_m \gamma_m}} - \varpi_{3,m} t Gr \\ &\geq c_m t (\varpi_{1,m} Gr^{\Delta_m} - \varpi_{3,m} Gr) \end{aligned} \quad (\text{III.11})$$

having used the assumed lower bound in the theorem and the upper bound of $\int_0^t D_1 d\tau$. Moreover, to have the dissipation greater than forcing requires $\Delta_m > 1$ so δ_m must lie in the range as in (III.9) because $1 < \Delta_m \leq 4$. For large Gr the negative Gr -term in (III.6) is dropped so the integral in the denominator of (III.6) is estimated as

$$\int_0^t \exp \left(-2 \int_0^\tau X_m d\tau' \right) d\tau \leq [2\tilde{c}_m \varpi_{1,m}]^{-1} Gr^{-\Delta_m} (1 - \exp [-2\varpi_{1,m} \tilde{c}_m t Gr^{\Delta_m}]), \quad (\text{III.12})$$

and so the denominator of (III.6) satisfies

$$\text{Denominator} \geq [D_m(0)]^{-2} - c_{2,m} c_{1,m} (2\tilde{c}_m)^{-1} Gr^{-\Delta_m} (1 - \exp [-2\varpi_{1,m} \tilde{c}_m t Gr^{\Delta_m}]). \quad (\text{III.13})$$

This can never go negative if $[D_m(0)]^{-2} > c_{1,m} c_{2,m} (2\tilde{c}_m)^{-1} Gr^{-\Delta_m}$, which means $D_m(0) < C_m Gr^{\frac{1}{2}\Delta_m}$. ■

B. A mechanism for intermittency

A major feature of intermittent flows lies in the strong, spiky, excursions of the vorticity away from averages with periods of relative inactivity between the spikes. How do these aperiodic cycles appear in solutions and does the critical lower bound imposed as an assumption in Theorem 4 lead to this? Using the average notation $\langle \cdot \rangle_t$, equation (III.6) shows that if $\langle D_m \rangle_t$ lies above critical then $D_m(t)$ collapses exponentially. Experimentally, signals go through cycles of growth and collapse so it is not realistic to expect this critical lower bound to hold for all time. .

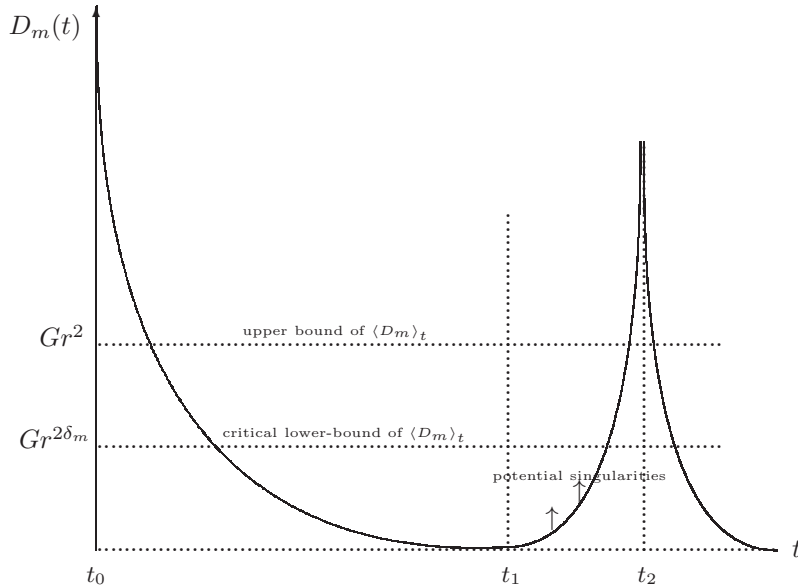


Figure 1: A cartoon of $D_m(t)$ versus t illustrating the four phases of intermittency. The vertical arrows depict the region where there is the potential for needle-like singular behaviour.

To understand intermittency we turn to Figure 1 and draw the horizontal line at $Gr^{2\delta_m}$ as the critical lower bound on $\langle D_m \rangle_t$. Within this allowed range, $D_m(t)$ will decay exponentially fast. Because integrals must take account of history, there will be a delay before $\langle D_m \rangle_t$ decreases below the value above which a zero in the denominator of (III.6) can be prevented (at t_1): at this point all constraints are removed and the pointwise solution $D_m(t)$ is free to grow rapidly again in the interval $t_1 \leq t \leq t_2$. If the value of this integral drops below critical then it is in this interval that the occurrence of singular events (depicted by vertical arrows)) must still formally be considered: see the discussion in §IV. Provided a solution still exists at this point, growth in D_m will be such that, after another delay, it will force $\langle D_m \rangle_t$ above critical and the system, with a re-set of initial conditions at t_2 , is free to go through another cycle akin to a relaxation oscillator.

IV. CONCLUSION

The results of Theorems II.2 and 4 are new. More appropriate for the unforced case, Theorem II.2 can be summarized thus: if there sufficient regions for which $Z_m > 1$ on the t -axis, such that

$$\int_0^t \ln \left(\frac{1 + Z_m}{c_{4,m}} \right) d\tau \geq 0 \quad Z_m = D_{m+1}/D_m \quad (\text{IV.1})$$

then $D_m(t) \leq D_m(0)$. In fact both of the regimes $Z_m \leq 1$ are physically realistic but numerical experiments may suggest which of these two regimes are the most manifest and whether a cross-over from one to the other occurs.

For the forced case, Theorem 4 can be summarized thus: if

$$\int_0^t D_m d\tau \geq c_m (t Gr^{2\delta_m} + \eta_2) \quad (\text{IV.2})$$

then $D_m(t)$ collapses exponentially. The main question is the meaning of the regime below this critical value

$$\int_0^t D_m d\tau < c_m (t Gr^{2\delta_m} + \eta_2) . \quad (\text{IV.3})$$

It might be natural to suppose, on intuitive grounds, that singular behaviour would be less likely to occur when the upper bound is smaller, yet this behaviour cannot be wholly ruled out. What can be ruled out are potentially singular spikes that substantially contribute to the time integral of $D_m(t)$ because they would push it over its critical value, thereby forcing exponential collapse in $D_m(t)$. However, there still remains the possibility of needle-like singular spikes (depicted by arrows in Figure 1) that contribute little or nothing to the time integral of $D_m(t)$ in (IV.3). It has been shown that the time-axis can potentially be divided into ‘good’ and ‘bad’ intervals, the name of this second set implying that no control over solutions has yet been found [38]. To summarize the argument in reference [38], it is very easy to show that for an arbitrary set of parameters $0 < \mu_m < 1$,

$$\int_0^t \left(\left[\frac{D_{m+1}}{D_m} \right]^{\frac{1-\mu_m}{\mu_m}} - [c_m^{-1} Gr^{-2} D_{m+1}^{\mu_m}]^{\frac{1-\mu_m}{\mu_m}} \right) d\tau \geq 0 . \quad (\text{IV.4})$$

Thus there are potentially ‘bad’ intervals of the t -axis on which

$$\frac{D_{m+1}}{D_m} \geq c_m^{-1} Gr^{-2} D_{m+1}^{\mu_m} \quad (\text{IV.5})$$

but on which no upper bounds have been found. Using the fact that $\Omega_{m+1} \geq \Omega_m$, it follows that on these intervals

$$D_{m+1} \geq c_m Gr^{\frac{2}{1+\mu_m-\alpha_{m+1}/\alpha_m}} . \quad (\text{IV.6})$$

For large m reduces to $D_{m+1} \geq c_m Gr^{2/\mu_m}$. Given that μ_m could be chosen very small these lower bounds could be very large indeed, incidentally too large for the Navier-Stokes equations to remain valid. It is possible that these are the root cause of the potential singularities discussed above and labelled by vertical arrows in Figure 1.

It is also possible to interpret this behaviour informally using the so-called β -model of Frisch, Sulem and Nelkin [40] who modelled a Richardson cascade by taking a vortex of scale $\ell_0 \equiv L$ which cascades into daughter vortices, each of scale ℓ_n . The vortex domain halves at each step: thus $\ell_0/\ell_n = 2^n$. The self-similarity dimension d is then introduced by considering the number of offspring at each step as 2^d , where d is formally allowed to take non-integer values. In

d dimensions the Kolmogorov scaling calculations for velocity, turn-over time and other variables have multiplicative factors proportional to $(\ell_0/\ell_n)^{(3-d)/3}$: see [32, 40]. Equating the turn-over and viscous times in the standard manner one arrives at (ℓ_d is their viscous dissipation length)

$$L\lambda_m^{-1} \equiv \ell_0/\ell_d \sim Re^{\frac{3}{d+1}}. \quad (\text{IV.7})$$

This gives the Kolmogorov inverse scale of $Re^{3/4}$ in a three-dimensional domain[43]. To interpret the meaning of (IV.3) in the light of (I.10) requires the conversion of (IV.8) into Reynolds number notation ($Gr \rightarrow Re^2$) as in [37]

$$\int_0^t D_m d\tau < c_m (t Re^{3\delta_m} + \eta_2). \quad (\text{IV.8})$$

with corresponding length scales

$$L\lambda_m^{-1} \lesssim Re^{\frac{3\delta_m}{2\alpha_m}}. \quad (\text{IV.9})$$

This makes $d_m = 2\alpha_m\delta_m^{-1} - 1$ where the range of δ_m is given in (III.9). In the large m limit the range of δ_m widens to $11/20 < \delta_m < 1$ implying that d_m lies in the range $0 < d_m < 9/11$. However, it is conceivable that the sharp result for the lower bound on δ_m is $1/2$ which alters the range to $0 < d_m < 1$. The upper bound of, or near, unity is consistent with the result of Caffarelli, Kohn and Nirenberg [41] who showed that the singular set of the three-dimensional Navier-Stokes equations in four-dimensional space-time has zero 1-dimensional Hausdorff measure. Thus, it is possible that singularities that make no contribute to the integral in (IV.8) may conceivably be related to the CKN singular set. Whether or not these are physically realisable is open to question.

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Appendix A: Proof of Theorem 2

The following proof has some important variations on that given in [38]. Consider $J_m = \int_V |\omega|^{2m} dV$ such that

$$\frac{1}{2m} \dot{J}_m = \int_V |\omega|^{2(m-1)} \omega \cdot \{ \nu \Delta \omega + \omega \cdot \nabla \mathbf{u} + \text{curl} \mathbf{f} \} dV. \quad (\text{A.1})$$

Bounds on the three constituent parts of (A.1) are dealt with in turn, culminating in a differential inequality for J_m .

a) *The Laplacian term*: Let $\phi = \omega^2 = \omega \cdot \omega$. Then

$$\begin{aligned} \int_V |\omega|^{2(m-1)} \omega \cdot \Delta \omega dV &= \int_V \phi^{m-1} \left\{ \Delta \left(\frac{1}{2} \phi \right) - |\nabla \omega|^2 \right\} dV \\ &\leq \int_V \phi^{m-1} \Delta \left(\frac{1}{2} \phi \right) dV. \end{aligned} \quad (\text{A.2})$$

Using the fact that $\Delta(\phi^m) = m \{ (m-1)\phi^{m-2} |\nabla \phi|^2 + \phi^{m-1} \Delta \phi \}$ we obtain

$$\begin{aligned} \int_V |\omega|^{2(m-1)} \omega \cdot \Delta \omega dV &\leq -\frac{1}{2}(m-1) \int_V \phi^{m-2} |\nabla \phi|^2 dV + \frac{1}{2m} \int_V \Delta(\phi^m) dV \\ &= -\frac{2(m-1)}{m^2} \int_V |\nabla(\omega^m)|^2 dV, \end{aligned} \quad (\text{A.3})$$

having used the Divergence Theorem. Thus we have

$$\int_V |\omega|^{2(m-1)} \omega \cdot \Delta \omega dV \leq \begin{cases} -\int_V |\nabla \omega|^2 dV & m = 1, \\ -\frac{2}{c_{1,m}} \int_V |\nabla A_m|^2 dV & m \geq 2. \end{cases} \quad (\text{A.4})$$

where $A_m = \omega^m$ and $\tilde{c}_{1,m} = m^2/(m-1)$ with equality at $m = 1$. The negativity of the right hand side of (A.4) is important. Both $\|\nabla A_m\|_2$ and $\|A_m\|_2$ will appear later in the proof.

b) *The nonlinear term in (A.1)*: After a Hölder inequality, the second term in (A.1) becomes

$$\begin{aligned} \int_{\mathcal{V}} |\omega|^{2m} |\nabla \mathbf{u}| dV &\leq \left(\int_{\mathcal{V}} |\nabla \mathbf{u}|^{2(m+1)} dV \right)^{\frac{1}{2(m+1)}} \left(\int_{\mathcal{V}} |\omega|^{2(m+1)} dV \right)^{\frac{m}{2(m+1)}} \left(\int_{\mathcal{V}} |\omega|^{2m} dV \right)^{\frac{1}{2}} \\ &\leq c_m \left(\int_{\mathcal{V}} |\omega|^{2(m+1)} dV \right)^{\frac{1}{2}} \left(\int_{\mathcal{V}} |\omega|^{2m} dV \right)^{\frac{1}{2}} \\ &= c_m J_{m+1}^{1/2} J_m^{1/2}, \end{aligned} \quad (\text{A.5})$$

where the inequality $\|\nabla \mathbf{u}\|_p \leq c_p \|\omega\|_p$ for $p \in (1, \infty)$ has been used, which is based on a Riesz transform: note the exclusion of the case $m = \infty$ where a logarithm of norms of derivatives is necessary [39] – see [42] for remarks on L^∞ -estimates. Together with (A.2) this makes (A.1) into

$$\frac{1}{2m} J_m \leq -\frac{\nu}{\tilde{c}_{1,m}} \int_{\mathcal{V}} |\nabla(\omega^m)|^2 dV + c_m J_{m+1}^{1/2} J_m^{1/2} + \int_{\mathcal{V}} |\omega|^{2(m-1)} \omega \cdot \text{curl} \mathbf{f} dV. \quad (\text{A.6})$$

c) *The forcing term in (A.1)*: Now we use the smallest scale in the forcing ℓ with $\ell = L/2\pi$ to estimate the last term in (A.6)

$$\int_{\mathcal{V}} |\omega|^{2(m-1)} \omega \cdot \text{curl} \mathbf{f} dV \leq \|\omega\|_{2m}^{2m-1} \|\nabla \mathbf{f}\|_{2m} \quad (\text{A.7})$$

However, by going up to at least $N \geq 3$ derivatives in a Sobolev inequality and using our restriction of single-scale forcing at $k \sim \ell^{-1}$ (with $\ell = L/2\pi$) it can easily be shown that $\|\nabla \mathbf{f}\|_{2m} \leq c \|\mathbf{f}\|_2 \ell^{\frac{3-5m}{2m}}$ and so

$$\left| \int_{\mathcal{V}} |\omega|^{2(m-1)} \omega \cdot \text{curl} \mathbf{f} dV \right| \leq c \Omega_m^{2m-1} L^3 \varpi_0^2 Gr. \quad (\text{A.8})$$

d) *A differential inequality for J_m* : Recalling that $A_m = \omega^m$ allows us to re-write J_{m+1} as

$$J_{m+1} = \|A_m\|_{2(m+1)/m}^{2(m+1)/m}. \quad (\text{A.9})$$

A Gagliardo-Nirenberg inequality yields

$$\|A_m\|_{2(m+1)/m} \leq c_m \|\nabla A_m\|_2^{3/2(m+1)} \|A_m\|_2^{(2m-1)/2(m+1)} \quad (\text{A.10})$$

which means that

$$J_{m+1} \leq c_m \left(\int_{\mathcal{V}} |\nabla(\omega^m)|^2 dV \right)^{3/2m} J_m^{(2m-1)/2m}. \quad (\text{A.11})$$

With the definition

$$\beta_m = \frac{4}{3}m(m+1) \quad (\text{A.12})$$

(the factor of $\frac{4}{3}$ is different from that in [38]), (A.11) can be used to form Ω_{m+1}

$$\begin{aligned} \Omega_{m+1} &= \left(L^{-3} J_{m+1} + \varpi_0^{2(m+1)} \right)^{1/2(m+1)} \leq c_m \left(L^{-1} \int_{\mathcal{V}} |\nabla(\omega^m)|^2 dV + \varpi_0^{2m} \right)^{1/\beta_m} \\ &\quad \times \left[(L^{-3} J_m)^{1/2m} + \varpi_0 \right]^{(2m-1)/2(m+1)} \end{aligned} \quad (\text{A.13})$$

which converts to

$$c_m \left(L^{-1} \int_{\mathcal{V}} |\nabla(\omega^m)|^2 dV + \varpi_0^{2m} \right) \geq \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{\beta_m} \Omega_m^{2m}. \quad (\text{A.14})$$

This motivates us to re-write (A.6) as

$$\begin{aligned} \frac{1}{2m}(L^{-3}J_m) &\leq -\frac{\varpi_0}{\tilde{c}_{1,m}} \left(L^{-1} \int_{\mathcal{V}} |\nabla(\omega^m)|^2 dV \right) + \tilde{c}_{2,m} (L^{-3}J_{m+1})^{1/2} (L^{-3}J_m)^{1/2} \\ &\quad + c_{3,m} \varpi_0^2 \Omega_m^{2m-1} Gr. \end{aligned} \quad (\text{A.15})$$

Converting the J_m into Ω_m and using the fact that $\Omega_m \geq \varpi_0$

$$\dot{\Omega}_m \leq \Omega_m \left\{ -\frac{\varpi_0}{c_{4,m}} \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{4m(m+1)/3} + c_{5,m} \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{m+1} \Omega_m + c_{6,m} \varpi_0 Gr \right\} \quad (\text{A.16})$$

Using a Hölder inequality on the central term on the right hand side, with the definition of β_m given in (A.12), (A.16) finally becomes

$$\dot{\Omega}_m \leq \varpi_0 \Omega_m \left\{ -\frac{1}{c_{1,m}} \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{\beta_m} + c_{2,m} (\varpi_0^{-1} \Omega_m)^{2\alpha_m} + c_{3,m} Gr \right\}. \quad (\text{A.17})$$

Given the definition $D_m = (\varpi_0^{-1} \Omega_m)^{\alpha_m}$ it is found that

$$\left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{\beta_m} = \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^2 \quad (\text{A.18})$$

having used the fact that

$$\left(\frac{1}{\alpha_{m+1}} - \frac{1}{\alpha_m} \right) \beta_m = 2 \quad (\text{A.19})$$

and $\rho_m = \beta_m / \alpha_{m+1}$. This converts (A.17) to (I.13) of the theorem with $m = \infty$ excluded. The constants $c_{n,m}$ for $n = 1, 2, 3$ grow algebraically with m . ■

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